

Local Oscillator Limited Frequency Stability for Passive Atomic Frequency Standards Using Square Wave Frequency Modulation

Charles A. Greenhall, *Senior Member, IEEE*, and G. John Dick

Abstract—Atomic frequency standards using square wave frequency modulation effectively interrogate the atomic line by switching back and forth between two frequencies with equal atomic absorption values. For a symmetric absorption line, the slope of the responses will also be equal. In the quasistatic limit, this would seem to be an ideal interrogation process: the sign reversal of frequency slope can be removed by detection electronics to give an essentially unvarying sensitivity to local oscillator frequency variations. Such an interrogation would seem to eliminate L.O. aliasing and relieve stringent requirements on L.O. phase noise. Nevertheless, sign changes in the interrogation and detection processes mean that the sensitivity is actually zero at some point in the cycle. We derive consequences of this fact by an analysis in terms of the sensitivity function $g(t)$. For white phase noise, we derive an optimal form for $g(t)$ and show that the aliased noise always diverges as $g(t)$ approaches a constant. For flicker phase noise, we find a limiting form that could, in principle, eliminate the aliasing effect; in practice, however, the improvement is limited by a slow dependence on available bandwidth. Finally, we derive optimized forms for any phase noise spectrum.

I. INTRODUCTION

LIMITATIONS to passive frequency standards performance by aliased local oscillator fluctuations have previously been analyzed for both pulsed and continuous interrogation methods. Until recently, frequency standards using continuous frequency modulation interrogation, such as cesium beam tubes and rubidium gas cells, have been analyzed in terms of the Fourier sine coefficients of the modulation and demodulation waveforms. On the other hand, the analysis of pulse-mode standards, such as the linear ion trap and cesium fountain, is carried out in terms of a sensitivity function $g(t)$ that is derived in the time domain by a quantum mechanical analysis of the atomic interrogation [1]–[4]. The aliasing effect is expressed as a function of the Fourier cosine coefficients of $g(t)$. Here, we present an application of the sensitivity function methodology to a limiting case of square wave frequency modulation for passive frequency standards interrogated by continuous FM and treated in the quasistatic limit.

The effectiveness of square wave modulation or demodulation has been shown in recent work to be surprisingly smaller than might be expected. Venot *et al.* [5] (see [6]

also) find only minimal ($\approx 1\%$) performance improvement by including successively higher numbers of optimized harmonics in the detection waveform for the case of white L.O. phase noise. The surprise is that, mathematically, a constant sensitivity can be approached by increasing harmonic content, and a constant (unvarying) sensitivity would not cause any aliasing at all.

On the other hand, we should perhaps not be surprised to find that the actual limiting value of the aliased noise for square wave frequency modulation and detection is not zero. For systems with finite bandwidth, both the time-varying sensitivity of the atomic discriminator and the transfer function of the modulator must in fact cross zero while reversing sign. Thus, it makes sense to consider a general continuous form of the sensitivity $g(t)$ that goes from 0 to a positive value and back to 0 during half a cycle of the square wave and to study the limiting behavior of the aliased noise as the graph of $g(t)$ approaches a rectangle. For this form, which always includes zeros, it is not obvious that the aliasing effect is small.

In the following sections, we show how previous analyses of continuous FM interrogation can be put into the sensitivity-function setting, and we specify the mathematical framework used in the remainder of the paper. We investigate the behavior of the aliasing effect for white and flicker phase noise processes in terms of a dimensionless variance σ_g^2 that measures how much $g(t)$ differs from a constant. Working from explicit time dependencies for $g(t)$, as well as from Fourier coefficients, we calculate the zero frequency spectral density of aliased frequency variations for the locked local oscillator. For atomic frequency standards, this provides a lower bound for white frequency noise in the locked L.O. For white L.O. phase noise, we prove that this aliased noise spectral density diverges as $1/\sigma_g^2$ as $\sigma_g^2 \rightarrow 0$. We also derive the optimal shape of $g(t)$, a sequence of parabolic arches that reduces the aliasing effect from our nominal case by a factor of $6/\pi^2$. The sine coefficients of the corresponding demodulation waveform agree with those obtained previously [5]. Nevertheless, the aliasing effect for the optimal case is only 1.5% smaller than that for square wave modulation with sine wave demodulation ($\pi^2/16$), explaining why improvements to that widely used strategy have proven so hard to come by. For flicker phase noise, we present a form for $g(t)$ for which the aliasing effect approaches zero with decreasing σ_g^2 . Finally, we show how to obtain optimized sensitivity functions for arbitrary phase noise spectra.

Manuscript received January 6, 2000; accepted March 13, 2000.

The authors are with the Jet Propulsion Laboratory (CIT), Pasadena, CA 91109 (e-mail: charles.greenhall@jpl.nasa.gov).

II. THE SENSITIVITY FUNCTION

A. Previous Work

The sensitivity function originated as a tool for studying the L.O. noise aliasing effect in passive atomic frequency standards with pulsed interrogation [1]. In these standards, the detected frequency error is available only at multiples of a cycle period T_c . The error signal at time nT_c has the form

$$\int_{(n-1)T_c}^{nT_c} g(t) y(t) dt \quad (1)$$

where $y(t)$ is the instantaneous normalized L.O. frequency error. The sensitivity function $g(t)$, which has been derived for different interrogation methods by quantum mechanical computations, is a T_c -periodic function with Fourier series

$$g(t) = \sum_{n=-\infty}^{\infty} g_n e^{i2n\pi\nu_c t} \quad (2)$$

where $\nu_c = 1/T_c$. (The meaning of g_n is different from the one in [7].) When the error signal (1) is used in a frequency control loop, the L.O. frequency noise near the cycle frequency ν_c and its harmonics is aliased into the loop passband by the Fourier components of the periodic function $g(t)$, causing a spurious random low frequency correction voltage to be delivered to the L.O. This effect induces a noise component in the controlled L.O. frequency inside the loop passband; the spectral density of this induced noise at Fourier frequency zero is given by

$$S_y(0) = \frac{2}{g_0^2} \sum_{n=1}^{\infty} |g_n|^2 S_y^{\text{LO}}(n\nu_c) \quad (3)$$

where $S_y^{\text{LO}}(\nu)$ is the spectrum of the normalized frequency of the uncontrolled L.O. This noise appears as an Allan deviation component

$$\sigma_y(\tau) = \sqrt{\frac{S_y(0)}{2\tau}} \quad (4)$$

in the output of the controlled L.O. for τ greater than the response time of the loop. Control loop analyses leading to (3) from different points of view are available in the literature [1], [2], [8]–[10].

The present paper deals with a situation in which the L.O. frequency error $y(t)$ is detected continuously. Let $M(t)$ and $D(t)$ be the synchronous modulation and demodulation waveforms, assumed to be periodic with period T_m , odd about 0, and even about $T_m/4$. In the quasistatic approximation, the error signal, linearized with respect to small fluctuations $y(t)$, is $D(t)M(t)y(t)$. Under these conditions, the function $g(t) = D(t)M(t)$ is periodic with period $T_c = T_m/2$ and even about 0 (and about $T_c/2$); thus, $g(t)$ will be called the sensitivity function for this detection scheme. Fig. 1 shows an example in which $M(t)$ is a square wave, $D(t)$ is a sine wave, and $g(t)$ is a rectified sine wave.

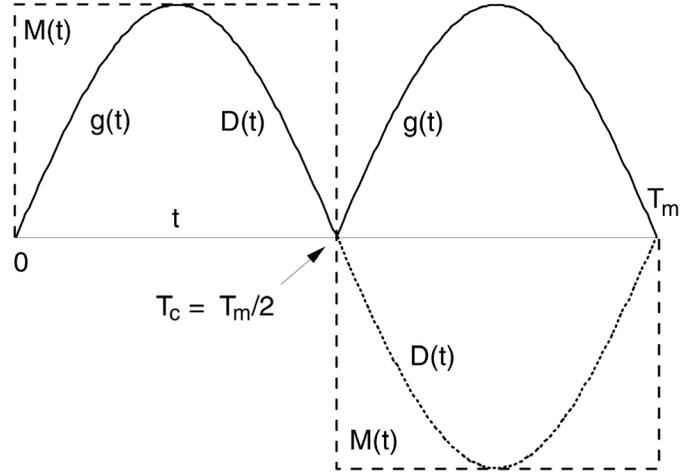


Fig. 1. Square wave modulation $M(t)$ and sine wave demodulation $D(t)$, both with period T_m , combine to give a rectified sine-wave sensitivity function $g(t)$ with period $T_c = T_m/2$.

Passive frequency standards with this kind of error signal have been analyzed in the literature for different cases of $M(t)$ and $D(t)$. In all cases (unless $M(t)$ and $D(t)$ are both perfect square waves), there is an aliasing effect analogous to the one for pulsed interrogation, and (3) holds. These analyses, after expanding $M(t)$ or $D(t)$ as a sine series

$$2 \sum_{n=0}^{\infty} C_{2n+1} \sin((2n+1)\pi\nu_c t), \quad (5)$$

calculate the aliasing effect in terms of the sine coefficients C_{2n+1} without mentioning $g(t)$ explicitly; nevertheless, formulas for its Fourier coefficients g_n can be seen in the results, which are all of form (3). We summarize three of these analyses, giving the g_n formula for each.

Audoin *et al.* [11] considered a modulation waveform $M(t)$ that depended on the applied modulation and the atomic response curve, along with sine wave demodulation $D(t) = 2 \sin(\pi\nu_c t)$. If $M(t)$ is expanded as (5), then $g(t) = 2M(t) \sin(\pi\nu_c t)$ has Fourier coefficients

$$g_0 = 2C_1, \quad g_n = C_{2n+1} - C_{2n-1} \text{ for } n \neq 0. \quad (6)$$

Venot *et al.* [5] considered square wave modulation and a general demodulation waveform $D(t)$ given by (5). In this case, we can assume that $g(t) = D(t)$ for $0 \leq t \leq T_c$; thus, on this interval, $g(t)$ has the two orthogonal expansions:

$$\begin{aligned} g(t) &= g_0 + 2 \sum_{n=1}^{\infty} g_n \cos(2n\pi\nu_c t) \\ &= 2 \sum_{n=0}^{\infty} C_{2n+1} \sin((2n+1)\pi\nu_c t). \end{aligned} \quad (7)$$

Taking the inner product of both expansions with $\cos(2n\pi\nu_c t)$, we find that

$$g_n = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{C_{2k+1} (2k+1)}{(2k+1)^2 - 4n^2} \text{ for all } n. \quad (8)$$

By an analysis in the frequency domain, De Marchi *et al.* [12] obtained a result equivalent to (8). They assumed square wave modulation and demodulation but with a low-pass filter h_F applied between them to represent bandwidth limitations of the system. It is useful to discuss a generalization of their setup. Suppose that the detected error signal is

$$B(t) \int h_F(t-u) M(u) y(u) du$$

where $M(t)$ and $B(t)$ are T_m -periodic waveforms. The filter h_F , which is not necessarily a lowpass filter, is assumed to have a frequency response $H_F(\nu)$ that is essentially constant within intervals $n\nu_m \pm w$, where w is the loop bandwidth. Although this error signal is not of form $g(t)y(t)$, an analysis similar to the one in [12] shows that, as far as the aliasing effect is concerned, this error signal is mathematically equivalent to

$$\left[\int B(u)h_F(u-t)du \right] M(t)y(t) = D(t)M(t)y(t) .$$

Even though the filter $h_F^-(t) = h_F(-t)$ applied to $B(t)$ may be noncausal, its output $D(t)$ is a well-defined T_m -periodic function in this steady-state analysis. If $M(t + \frac{1}{2}T_m) = -M(t)$ and $D(t + \frac{1}{2}T_m) = -D(t)$, then the function $g(t) = D(t)M(t)$ has period $\frac{1}{2}T_m$. Thus, $g(t)$ deserves to be interpreted as the sensitivity function for the detection. In the case treated in [12], $M(t)$ and $B(t)$ are both square waves with amplitude 1, and $h_F(t)$ is an even lowpass filter; then, because the filtered square wave $D(t)$ has an expansion of form (5), we see that (8) holds with

$$C_{2n+1} = \frac{2}{\pi} \frac{H_F((2n+1)\nu_m)}{2n+1} .$$

This analysis also illuminates the practice of achieving sine wave demodulation by applying a lowpass or bandpass filter before the demodulation [13]. For example, we could let $h_F(t) = e^{-\lambda t} \sin(2\pi\nu_m t)$ for $t \geq 0$, where λ is chosen to make the bandwidth intermediate between the loop bandwidth and ν_m . Then, h_F^- picks off mostly the fundamental component of $B(t)$, which could be a square wave; thus, $D(t)$ is essentially a sine wave. A time delay must be provided to make $D(t)$ line up properly with $M(t)$.

B. Present Approach

Although we shall keep the factors $M(t)$ and $D(t)$ in mind, it is reasonable to investigate how the aliasing effect depends on their product $g(t)$, regarded as an independent object of study. For convenience, we set $T_c = 1$, and we shall drop the assumption that $g(t)$ is even. There is precedent in the literature for doing so. Lemonde *et al.* [4] and Makdissi *et al.* [14] derived asymmetric sensitivity functions for periodic and continuous interrogation disciplines; Barillet *et al.* [15] allowed cosine terms in the expansion of $D(t)$. Also, we wish to leave open the possibility that a

dynamic analysis might lead to an asymmetric $M(t)$. Consequently, we must allow the coefficients g_n to be complex, satisfying $g_{-n} = \overline{g_n}$. Nevertheless, all of the examples and optimization solutions to follow are even functions.

Let us place the following mathematical conditions on $g(t)$, $0 \leq t \leq 1$:

- its Fourier coefficients

$$g_n = \int_0^1 e^{-i2n\pi t} g(t) dt \tag{9}$$

satisfy

$$\sum_{n=-\infty}^{\infty} |g_n| < \infty ; \tag{10}$$

- $g(0) = 0$;
- $g_0 = \int_0^1 g(t)dt > 0$.

The first condition says that the Fourier series (2) converges absolutely and uniformly; this forces $g(t)$ to extend to a continuous function on the real line with period 1. The second condition says that either the modulation or the demodulation waveform has to pass continuously through zero, even though the other might be an ideal square wave. Because of the first condition, $g(1) = 0$ also. The third condition is needed so that $g(t)y(t)$ can be a reasonable error signal on average; the open-loop gain contains g_0 as a factor.

A real-valued function $g(t)$ that fulfills these conditions will be called an admissible sensitivity. Two admissible sensitivities that are proportional to each other are equivalent for our purposes. For an admissible sensitivity, (2) and the second condition yield

$$g_0 = - \sum_{n \neq 0} g_n . \tag{11}$$

To measure how much $g(t)$ differs from a constant sensitivity, we define its normalized variance σ_g^2 by

$$\sigma_g^2 = \frac{1}{g_0^2} \int_0^1 [g(t) - g_0]^2 dt = \frac{1}{g_0^2} \int_0^1 g^2(t) dt - 1 . \tag{12}$$

Then

$$\sigma_g^2 = \frac{2}{g_0^2} \sum_{n=1}^{\infty} |g_n|^2 . \tag{13}$$

A previous study [7] used a different measure that applied to a class of sensitivity functions that are even about $\frac{1}{2}$ and increasing on $[0, \frac{1}{2}]$.

To make a dimensionless version of the aliasing formula (3), let us normalize it by the case of sine wave modulation and demodulation (sine \times sine detection). For this case, $g(t) = \sin^2(\pi t) = \frac{1}{2}(1 - \cos(2\pi t))$ (see Fig. 2), so that $g_0 = \frac{1}{2}$, $g_1 = g_{-1} = -\frac{1}{4}$, and the other g_n are zero. Substituting into (3), we obtain the aliasing formula

$$S_y(0; \sin^2) = \frac{1}{2} S_y^{LO}(\nu_c) . \tag{14}$$

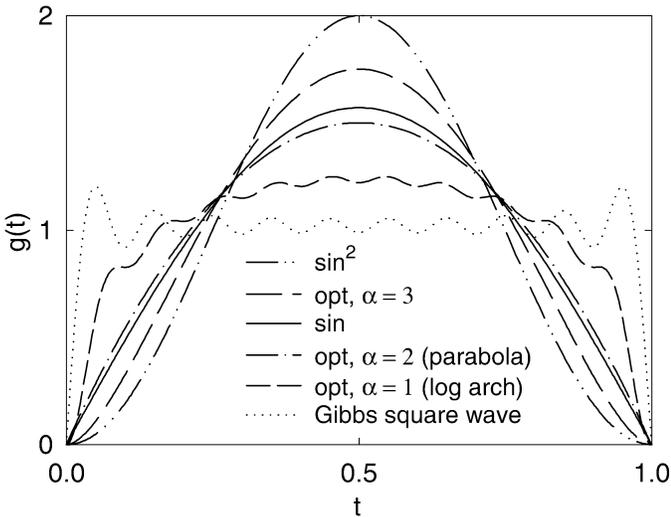


Fig. 2. A collection of sensitivity functions, normalized to have a mean value $g_0 = 1$. Three of the functions are optimized for $\nu^{\alpha-2}$ phase noise. The logarithmic arch is the optimal sensitivity of form $\sum_{n=-10}^{10} g_n e^{i2n\pi t}$ for flicker phase noise ($\alpha = 1$). The Gibbs square wave is a partial sum (10 sine terms) of the Fourier series of a square wave.

Assume $S_y^{\text{LO}}(\nu_c) > 0$. Then, dividing (3) by (14), we can define a normalized aliasing factor by

$$A = \frac{4}{g_0^2} \sum_{n=1}^{\infty} |g_n|^2 \frac{S_y^{\text{LO}}(n\nu_c)}{S_y^{\text{LO}}(\nu_c)}. \quad (15)$$

It will be shown later (Section V) that we can always make $A < 1$ by a good choice of $g(t)$; in other words, sine \times sine detection can always be improved upon, in principle.

Our purpose is to study the behavior of A for different types of L.O. noise. For a power-law noise spectrum, $S_y^{\text{LO}}(\nu) = h_\alpha \nu^\alpha$, we write $A = A_\alpha$, where

$$A_\alpha = \frac{4}{g_0^2} \sum_{n=1}^{\infty} |g_n|^2 n^\alpha. \quad (16)$$

For $\alpha = 0$ (white frequency), we have $A_0 = 2\sigma_g^2$; the aliasing factor always tends to zero as g approaches a constant sensitivity. Let us look at the more interesting cases $\alpha = 2$ (white phase) and $\alpha = 1$ (flicker phase).

III. WHITE PHASE

A. Optimal Sensitivity Function

For white phase noise, we show that the aliasing factor

$$A_2 = \frac{4}{g_0^2} \sum_{n=1}^{\infty} |g_n|^2 n^2 \quad (17)$$

has a positive minimum over all admissible g , and we derive the optimal $g(t)$. If $A_2 < \infty$ for an admissible $g(t)$,

then $g(t)$ is absolutely continuous and has an L^2 derivative $g'(t)$, whose n th Fourier coefficient is $i2\pi n g_n$. Because the L^2 norm of $g'(t)$ is given by

$$\|g'\|^2 = \int_0^1 |g'(t)|^2 dt = 8\pi^2 \sum_{n=1}^{\infty} |n g_n|^2,$$

we have

$$A_2 = \frac{\|g'\|^2}{2\pi^2 g_0^2}. \quad (18)$$

Knowing that $0 = g(0) = g(1) = \int_0^1 g'(t) dt$, we can express g_0 in terms of $g'(t)$ as follows:

$$\begin{aligned} g_0 &= \int_0^1 g(u) du = \int_0^1 \int_0^u g'(t) dt du \\ &= \int_0^1 g'(t) (1-t) dt = \int_0^1 g'(t) \left(\frac{1}{2} - t\right) dt. \end{aligned}$$

Thus, letting $f(t) = \frac{1}{2} - t$, we can rewrite (18) as

$$\frac{1}{A_2} = 2\pi^2 \left\langle \frac{g'}{\|g'\|}, f \right\rangle^2 \quad (19)$$

in terms of the L^2 inner product $\langle f_1, f_2 \rangle = \int_0^1 f_1 \overline{f_2}$. The quantity A_2^{-1} can be maximized by making the unit vector $g'/\|g'\|$ point in the direction of f ; we are allowed to do that because $f(t)$ is itself the derivative of the admissible sensitivity $\frac{1}{2}t(1-t)$, which, therefore, has to be the optimal choice. Thus, the optimal sensitivity function for white phase noise is a parabolic arch. Setting $g' = f$ in (19) gives

$$\min A_2 = \frac{1}{2\pi^2 \|f\|^2} = \frac{6}{\pi^2} \approx 0.60793,$$

which can be attained with square wave modulation and a demodulation waveform $D(t)$ consisting of a series of alternating positive and negative parabolic arches. This result confirms that of Venot *et al.* [5], whose optimal demodulation coefficients $C_{2n+1} = (2n+1)^{-3}$ are indeed the sine coefficients of a parabolic arch wave.

Because a parabolic arch looks much like a sinusoidal arch (Fig. 2), it is interesting to compare the minimal aliasing factor to the aliasing factor for square wave modulation and sine wave demodulation (square \times sine detection). A straightforward evaluation of (18) for $g(t) = \sin(\pi t)$ gives $A_2 = \pi^2/16 \approx 0.61685$, which is greater than $\min A_2$ by a factor of $\pi^4/96 \approx 1.0147$. For white phase noise, square \times sine detection is almost optimal.

The optimal sensitivity can be derived by yet another method, which will be used again later. After substituting (11) into (17) and setting $x_n = n g_n$, $a_n = -1/n$ ($n \neq 0$), we have

$$\frac{1}{A_2} = \frac{\left(\sum_{n \neq 0} x_n a_n\right)^2}{2 \sum_{n \neq 0} |x_n|^2} = \frac{1}{2} \left\langle \frac{x}{\|x\|}, a \right\rangle^2$$

where we are now using the inner product, $\langle u, v \rangle = \sum_{n \neq 0} u_n \bar{v}_n$, and its norm, $\|u\| = \sqrt{\langle u, u \rangle}$, for two-sided complex sequences with the term for $n = 0$ omitted. For the optimal solution, set $x_n = a_n$, i.e., $g_n = -n^{-2}$ ($n \neq 0$), which is proportional to the n th Fourier coefficient of $t(1-t)$; then, $(\min A_2)^{-1} = \frac{1}{2} \|a\|^2 = \sum_{n=1}^{\infty} n^{-2} = \pi^2/6$ as before.

B. Comparison with Experiments

The aliasing factor $\pi^2/16$ for square \times sine detection says that there is only a 2.1-dB improvement in the aliasing effect when we go from sine \times sine detection to square \times sine detection. This result disagrees with the measurements of Deng *et al.* [13] on a rubidium cell. By injecting bandpass-filtered phase noise into the L.O. and observing the aliasing effect via its Allan variance, they obtained measurements of coefficients (C_{2n} in their notation) proportional to $n|g_n|$, averaged over ranges of n . Their results correspond to a decrease in the aliasing effect of approximately 20 dB. To the authors' knowledge, this large discrepancy has not been explained.

There is better agreement with recent measurements of Barillet *et al.* [15] on a cesium tube. They observed the aliasing effect by injecting controlled amounts of phase noise in the vicinity of frequencies $n\nu_c$, $1 \leq n \leq 5$. For square \times sine detection with white phase noise, they measured $\sigma_y(8 \text{ s}) = 7.3 \times 10^{-12}$; to calibrate our calculations, we chose the value of phase spectral density that made our calculated value of $\sigma_y(8 \text{ s})$ agree with the measured value. After attenuating the noise density at ν_c by 14 dB, they measured $\sigma_y(8 \text{ s}) = 3.9 \times 10^{-12}$; our calibrated calculations yielded 3.7×10^{-12} . Using the same noise conditions and square wave modulation, they introduced a demodulation waveform

$$D(t) = 2 [\sin(\pi\nu_c t) + C_3 \sin(3\pi\nu_c t)]$$

with the object of minimizing the aliasing effect with respect to C_3 . (Square \times sine detection corresponds to $C_3 = 0$.) Although the measured values of $\sigma_y(8 \text{ s})$ and our calculated values show a rough qualitative agreement for $0 \geq C_3 \geq -0.4$, the calculated values vary between 0.94 and 0.55 times the measured values.

C. Approach to a Rectangle

Suppose that the graph of the admissible sensitivity function on $[0,1]$, which passes through the points $(0,0)$ and $(1,0)$, approaches a rectangle in the sense that σ_g^2 tends to 0. Then, for white phase noise, the aliasing factor A_2 tends to infinity at least as fast as $1/\sigma_g^2$. More precisely,

$$A_2 \sigma_g^2 \geq \frac{1}{2\pi^2} \quad (20)$$

for all admissible $g(t)$; moreover, the constant in (20) is best possible.

To prove (20), let $f(t) = 1 - g(t)/g_0$. Then,

$$f(0) = f(1) = 1, \quad \int_0^1 f(t) dt = 0. \quad (21)$$

We may assume that $A_2 < \infty$. From (12) and (18), we have

$$\sigma_g^2 = \int_0^1 f(t)^2 dt, \quad A_2 = \frac{1}{2\pi^2} \int_0^1 f'(t)^2 dt;$$

thus, (20) is equivalent to

$$\int_0^1 f^2 \int_0^1 f'^2 \geq 1. \quad (22)$$

To prove (22), observe that the continuous function f assumes its mean value 0 at some point a in $(0,1)$. Because f is absolutely continuous, so is f^2 , implying that f^2 is the integral of its derivative $2ff'$. Then

$$\begin{aligned} 1 &= \frac{1}{2} [f(0)^2 - f(a)^2 + f(1)^2 - f(a)^2] \\ &= - \int_0^a ff' + \int_a^1 ff' \\ &\leq \int_0^a |ff'| + \int_a^1 |ff'| = \int_0^1 |ff'| \\ &\leq \left(\int_0^1 f^2 \int_0^1 f'^2 \right)^{1/2} \end{aligned}$$

by Schwarz's inequality. This proves (22).

To show that the constant $(2\pi)^{-1}$ in (20) cannot be improved upon, we show that (22) is not valid in general with a constant greater than 1 on the right side. To make Schwarz's inequality efficient, we try to make $|f'(t)|$ proportional to $|f(t)|$ and retain the conditions in (21). Let $f(t) = be^{-ct} - d$ for $0 \leq t \leq \frac{1}{2}$ and $f(t) = f(1-t)$ for $\frac{1}{2} \leq t \leq 1$. After solving for b and d as functions of c such that $f(0) = 1$, $\int_0^1 f = 0$, we find that

$$\int_0^1 f^2 \int_0^1 f'^2 = 1 + \frac{4}{c} + O\left(\frac{1}{c^2}\right) \text{ as } c \rightarrow \infty.$$

This quantity can be made as close to 1 as we choose by making c large. The sensitivity function $g(t)$ that corresponds to this example approaches a rectangle as $c \rightarrow \infty$.

IV. FLICKER PHASE

Previous work [7] suggested that as $g(t)$ approaches a rectangle in some reasonable way, the aliasing factor

$$A_1 = \frac{4}{g_0^2} \sum_{n=1}^{\infty} |g_n|^2 n \quad (23)$$

approaches a positive constant. This is true for the method of horizontal shrinking used in [7] for symmetric $g(t)$, in which one splits the graph of $g(t)$ down the middle, shrinks

both halves horizontally toward the vertical lines $t = 0$ and $t = 1$ while retaining their shapes, and bridges the gap by the constant $g(1/2)$.

For another example, taken from [12], let

$$D(t) = \frac{4}{\pi} \sum_{n=0}^{N-1} \frac{\sin((2n+1)\pi t)}{2n+1},$$

which is a partial sum of the Fourier series of a square wave $q(t)$ with period 2 and amplitude 1. (In terms of the lowpass filter referred to earlier, $H_F((2n+1)\nu_m) = 1$ if $0 \leq n \leq N-1$ and 0 otherwise.) This function will be called the N -term Gibbs square wave (shown in Fig. 2 for $N = 10$) after the Gibbs phenomenon for the Fourier series of functions with jump discontinuities. Letting $M(t) = q(t)$ makes $g(t)$ the even extension of $D(t)$. Plots of g_n as a function of n and N are given in [12]. For large N , the summations in (8) and (23) can be approximated by integrals that can be evaluated in terms of known functions. Also, because $g(t) = D(t)$ on $[0,1]$ and because (5) is an orthogonal expansion, σ_g^2 can be evaluated directly from (12). (In fact, $\sigma_g^2 = 1/g_0 - 1$ with this particular scaling.) We find that $\sigma_g^2 \sim 2/(\pi^2 N)$ as $N \rightarrow \infty$, and

$$\lim_{N \rightarrow \infty} A_1 = \frac{28\zeta(3)}{\pi^4} \approx 0.3455 \quad (24)$$

where $\zeta(p) = \sum_{n=1}^{\infty} n^{-p}$.

These findings suggest the conjecture that A_1 , similar to A_2 , is bounded below by some positive constant. To show that this conjecture is false, we optimize the Fourier series (2) under the assumption of restricted harmonic content. Fix N , and assume that $g_n = 0$ for $|n| > N$. Set $x_n = g_n \sqrt{|n|}$, $a_n = -1/\sqrt{|n|}$; from (23) and (11), we obtain

$$\frac{1}{A_1} = \frac{(\sum x_n a_n)^2}{2 \sum x_n^2} = \frac{1}{2} \left\langle \frac{x}{\|x\|}, a \right\rangle^2$$

where the summations are over $1 \leq |n| \leq N$. As before, to maximize A_1^{-1} we set $x = a$, i.e., $g_n = -1/|n|$, and obtain g_0 from (11). Write $L_N = \sum_{n=1}^N n^{-1}$. Then, $L_N = \ln N + \gamma + O(1/N)$ as $N \rightarrow \infty$, where γ is Euler's constant, and we have

$$A_1 = \frac{1}{L_N} \sim \frac{1}{\ln N},$$

$$\sigma_g^2 = \frac{1}{2L_N^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sim \frac{\pi^2}{12(\ln N)^2}.$$

Thus, both A_1 and σ_g^2 can be made as small as we choose by making N large. The actual sensitivity function,

$$g_N(t) = 2 \sum_{n=1}^N \frac{1 - \cos(2n\pi t)}{n} = 4 \sum_{n=1}^N \frac{\sin^2(n\pi t)}{n}, \quad (25)$$

will be called the N -term logarithmic arch because $\sum_{n=1}^{\infty} n^{-1} \cos(2n\pi t) = -\ln |\sin(\pi t)|$ for $0 < t < 1$ [16]; it is shown in Fig. 2 for $N = 10$. Because of the slow dependence of A_1 on N , the limitations of the quasistatic

model might prevent us from making A_1 as small as 0.1, for example. Nevertheless, when $N = 10$, A_1 is already less than the limiting value (24) for the Gibbs square wave.

V. ARBITRARY PHASE NOISE

The previous argument generalizes easily to any L.O. phase noise spectral density. Let

$$s_n = \frac{S_y^{\text{LO}}(|n|\nu_c)}{S_y^{\text{LO}}(\nu_c)}$$

for $|n| \geq 1$; for simplicity, assume all $s_n > 0$. To minimize (15) over admissible $g(t)$ whose Fourier coefficients g_n vanish for $|n| > N$, we need only set $g_n = -1/s_n$ for $1 \leq |n| \leq N$ and determine g_0 from (11). For a given N , the optimal sensitivity and minimal aliasing factor are given by

$$g_N(t) = 2 \sum_{n=1}^N \frac{1 - \cos(2n\pi t)}{s_n} = 4 \sum_{n=1}^N \frac{\sin^2(n\pi t)}{s_n}, \quad (26)$$

$$(\min A)_N = \left(\sum_{n=1}^N \frac{1}{s_n} \right)^{-1}. \quad (27)$$

Because $s_1 = 1$ we see that $(\min A)_N < 1$; in other words, we can always improve on sine \times sine detection in principle. On the other hand, other studies [5], [6], [17], [18] have suggested that the aliasing effect could be reduced by suppressing the L.O. phase noise severely at $\nu_c = 2\nu_m$ before the atomic resonance sees it. In this situation, the sensitivity $g(t) = \sin^2(\pi t)$ is nearly optimal because $s_1^{-1} = 1$ dominates the other s_n^{-1} . This explains another result of Venot *et al.* [5]. Assuming square wave modulation and white phase noise with noise suppression at ν_c , they compute optimal sine coefficients of the demodulation waveform. For three-harmonic optimization and complete noise suppression at ν_c , their values of C_3/C_1 and C_5/C_1 differ by only 1% from $-\frac{1}{5}$ and $-\frac{1}{35}$, which are the exact values from the expansion of $\sin^2(\pi t)$ as a sine series (5).

In a narrow mathematical sense, of course, there is nothing special about the term for $n = 1$ in (3); for any $m > 1$, the optimal detection also improves on detection using $g(t) = \sin^2(m\pi t)$, which selects only the noise at frequency $m\nu_c$.

We can distinguish two behaviors for large N . If $\sum s_n^{-1} = \infty$ (e.g., $s_n \sim n^\alpha$ with $\alpha \leq 1$), then A can be made as small as we choose by making N large, subject to the limitations of the quasistatic approximation; we have already treated the borderline case $\alpha = 1$. If $\sum s_n^{-1} < \infty$ (e.g., $s_n \sim n^\alpha$ with $\alpha > 1$), then $g_N(t)$ converges uniformly as $N \rightarrow \infty$ to an admissible sensitivity that is optimal for this noise spectrum, and $(\min A)_N$ converges to the positive minimum $(\sum_{n=1}^{\infty} s_n^{-1})^{-1}$ of the aliasing factor. For $\alpha = 2$, we found that the optimal $g(t)$ is a parabolic arch.

For the sake of curiosity, we plot the optimal $g(t)$ for $\alpha = 3$ in Fig. 2; this function has zero slope at $t = 0$ and 1.

After obtaining a sensitivity function $g(t)$ that is judged to be desirable, we can try to realize it as $M(t)D(t)$. If one of the factors is a square wave, then the other factor is $g(t)$ on $[0,1]$. If one of the factors is a sine wave, we can either express the other factor as $g(t)/\sin(\pi t)$ in the time domain or obtain its sine expansion (5) from g_n by the formula

$$C_{2n+1} = - \sum_{k=n+1}^{\infty} g_k, \quad (28)$$

which follows from (6) and (11). If $g(t)$ has a finite cosine expansion, then the sine expansion (5) is also finite.

VI. CONCLUSIONS

Using the quasistatic theory, we have shown how to exploit the sensitivity function as a tool for studying the aliasing effect that arises when an atomic resonator is probed by a frequency-modulated signal in a control loop, and we have seen that simplicity and insight can be gained by direct use of the sensitivity function in analyses. In particular, we have quantitatively examined the attractive and almost obvious idea that as a constant interrogation sensitivity is approached, the aliased L.O. noise must go away. This idea turns out to be false because the sensitivity must actually be zero at some point. In fact, a sensitivity that is almost constant has little time to get down to zero and, thereby, tends to excite high frequency noise components. This problem is especially severe for white phase noise; the aliased noise has an absolute minimum and grows as the reciprocal of the normalized sensitivity variance as the sensitivity approaches a constant. Even for flicker phase noise, the minimal aliased noise tends to zero only like $1/\log N$, where N is the maximum degree of the cosine components of the sensitivity function; thus, in a practical sense, the aliased noise cannot be made arbitrarily small. Nevertheless, for a known phase noise spectrum, the sensitivity function can, in principle, be tailored to achieve the least possible aliased noise for a prescribed N . Whether this is accomplished by tailoring the modulation waveform, the demodulation waveform, or both is a matter of practical design.

ACKNOWLEDGMENT

This work was carried out at the Jet Propulsion Laboratory, California Institute of Technology, under a contract with the National Aeronautics and Space Administration.

REFERENCES

- [1] G. J. Dick, "Local oscillator induced instabilities in trapped ion frequency standards," in *Proc. 19th PTTI Applications and Planning Mtg.*, Redondo Beach, CA, 1987, pp. 133–147.
- [2] G. J. Dick, J. Prestage, C. Greenhall, and L. Maleki, "Local oscillator induced degradation of medium-term stability in passive atomic frequency standards," in *Proc. 22nd PTTI Applications and Planning Mtg.*, Vienna, VA, 1990, pp. 487–508.
- [3] G. Santarelli, C. Audoin, A. Makdissi, P. Laurent, G. Dick, and A. Clairon, "Frequency stability degradation of an oscillator slaved to a periodically interrogated atomic resonator," *IEEE Trans. Ultrason., Ferroelect., Freq. Contr.*, vol. 45, no. 4, pp. 887–894, Jul. 1998.
- [4] P. Lemonde, G. Santarelli, P. Laurent, F. Pereira Dos Santos, A. Clairon, and C. Salomon, "The sensitivity function: a new tool for the evaluation of frequency shifts in atomic spectroscopy," in *Proc. IEEE 1998 Int. Freq. Contr. Symp.*, Pasadena, CA, pp. 110–115.
- [5] D. Venot, R. Barillet, and C. Audoin, "On a possibility of reducing the frequency stability limitation by the interrogation noise in passive frequency standards," in *Proc. 8th Eur. Time Freq. Forum*, München, Germany, 1994, pp. 656–670.
- [6] R. Barillet, D. Venot, and C. Audoin, "Reducing the effect of interrogation phase noise on the frequency stability in passive frequency standards: experimental results and new investigation means," in *Proc. 1995 Int. Freq. Contr. Symp.*, San Francisco, CA, pp. 178–184.
- [7] G. J. Dick and C. Greenhall, "L.O. limited frequency stability for passive atomic frequency standards using 'square wave' frequency modulation," in *Proc. IEEE 1998 Int. Freq. Contr. Symp.*, Pasadena, CA, pp. 99–103.
- [8] C. Audoin, G. Santarelli, A. Makdissi, and A. Clairon, "Properties of an oscillator slaved to a periodically interrogated atomic resonator," *IEEE Trans. Ultrason., Ferroelect., Freq. Contr.*, vol. 45, no. 4, pp. 877–886, Jul. 1998.
- [9] C. Greenhall, "A derivation of the long-term degradation of a pulsed atomic frequency standard from a control-loop model," *IEEE Trans. Ultrason., Ferroelect., Freq. Contr.*, vol. 45, no. 4, pp. 895–898, Jul. 1998.
- [10] L. Lo Presti, D. Rovera, and A. De Marchi, "A simple analysis of the Dick effect in terms of phase noise spectral densities," *IEEE Trans. Ultrason., Ferroelect., Freq. Contr.*, vol. 45, no. 4, pp. 899–905, Jul. 1998.
- [11] C. Audoin, V. Candelier, and N. Dimarcq, "A limit to the frequency stability of passive frequency standards due to an inter-modulation effect," *IEEE Trans. Instrum. Meas.*, vol. 40, no. 2, pp. 121–125, Apr. 1991.
- [12] A. De Marchi, L. Lo Presti, and G. Rovera, "Square wave frequency modulation as a mean to reduce the effect of local oscillator instabilities in alkali vapor passive frequency standards," in *Proc. IEEE 1998 Int. Freq. Contr. Symp.*, Pasadena, CA, pp. 104–109.
- [13] J. Q. Deng, A. De Marchi, F. L. Walls, and R. E. Drullinger, "Frequency stability of cell-based passive frequency standards: reducing the effects of local-oscillator PM noise," in *Proc. IEEE 1998 Int. Freq. Contr. Symp.*, Pasadena, CA, pp. 95–98.
- [14] A. Makdissi, E. de Clerq, C. Audoin, and A. Clairon, "Generalization of the Dick effect to continuous beam frequency standards," in *Proc. IEEE 1997 Int. Freq. Contr. Symp.*, Orlando, FL, pp. 263–269.
- [15] R. Barillet, F. Hamouda, D. Venot, and C. Audoin, "Limitation of the frequency stability by local oscillator phase noise: new investigations," in *13th Eur. Freq. Time Forum and 1999 IEEE Int. Freq. Contr. Symp.*, Besançon, France, pp. 3–6.
- [16] A. Zygmund, *Trigonometric Series*. London: Cambridge Univ. Press, 1959.
- [17] J. Lowe, F. Walls, and R. Drullinger, "Ultra-high stability synthesizer for diode laser pumped rubidium," in *Proc. 1992 IEEE Freq. Contr. Symp.*, Hershey, PA, pp. 183–187.
- [18] C. Szekely, F. Walls, J. Lowe, R. Drullinger, and A. Novick, "Reducing local oscillator phase noise limitations on the frequency stability of passive frequency standards: tests of a new concept," *IEEE Trans. Ultrason., Ferroelect., Freq. Contr.*, vol. 41, no. 4, pp. 518–521, Jul. 1994.



Charles A. Greenhall (M'92-SM'00) was born in 1939 in New York City. He received a B.A. in physics in 1961 from Pomona College and a Ph.D. in mathematics in 1966 from Caltech; his dissertation was on Fourier series. For five years, he was an assistant professor of mathematics at the University of Southern California. Since 1981, he has worked at Jet Propulsion Laboratory in the field of clock statistics and measurement. He has furthered the theory of random processes with stationary n th increments, thereby obtaining results

for bias and confidence of frequency-stability and drift estimators. He invented the picket fence method for collecting zero dead-time data from interval counters and wrote the signal-processing software for a stability analyzer based on A-D conversion. Among his recent interests are wavelets for clock applications and development of a dual-mixer measurement system.



G. John Dick was born in 1939 in Ohio, received his A.B. in physics and mathematics from Bethel College (Kansas) in 1961, and received a Ph.D. degree in physics from U.C. Berkeley in 1969. After a postdoctoral fellowship at Caltech, he joined the research faculty there as Senior Research Fellow, developing RF resonator technology for superconducting accelerator applications. During 1979 and 1980, he was also appointed as a visiting faculty member at SUNY, Stonybrook. In 1986, he joined the staff of the Jet Propulsion

Laboratory, where he is presently Principal Member of the Technical Staff. Duties include development of cryogenic sapphire oscillators (10K CSO) presently under construction for installation in NASA's Deep Space Network and Project Scientist responsibility for SUMO, a Superconducting Microwave Oscillator experiment scheduled for the second flight of the Low Temperature Microgravity Physics Facility (LTMPF) aboard the International Space Station. Research interests include high-Q cryogenic resonator design and atomic frequency standard systematics. In April 1999, he was awarded "Le Prix Européen Temps-Fréquence", which was presented at the joint meeting of the 13th European Frequency and Time Forum and 1999 IEEE International Frequency Control Symposium. The award is granted every two years by the Société Française des Microtechniques et de Chronométrie to an individual or group for "degree of initiative and creativity, quality of work, degree of success obtained, and worldwide impact on the time and frequency associated community."